JOURNAL OF APPROXIMATION THEORY 43, 190–193 (1985)

Solution of Turán's Problem on Divergence of Lagrange Interpolation in L^p with $p>2$*

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Communicated by T. J. Rivlin

Received February 22, 1984

Existence of weight functions for which the Lagrange interpolating polynomials associated with the zeros of the corresponding orthogonal polynomials diverge in every L^p space with $p>2$ for some continuous function is proved.

Paul Turán [4, Problem VIII, p. 32] asked whether there exists a weight function w on $[-1, 1]$ such that, for some continuous function f, the corresponding Lagrange interpolating polynomials $L_n(w, f)$ satisfy

$$\limsup_{n\to\infty} \int_{-1}^{1} |f - L_n(w, f)|^p\, w = \infty \tag{1}$$

for every $p>2$. A weaker version of this problem is whether w exists so that, for every $p>2$, (1) holds with $f=f_p$ [4, Problem IX, p. 33]. We proved the existence of such a weight w in [3] and thus solved Problem IX of [4]. The purpose of this note is to prove a general result on weighted L^p divergence of Lagrange interpolation which implies an affirmative answer to Turán's Problem VIII.

Let $d\alpha$ be a positive measure supported in $[-1, 1]$ such that $\operatorname{supp}(d\alpha)$ is an infinite set and let $\{x_{kn}(d\alpha)\}_{k=1}^{n}$ denote the zeros of the corresponding nth-degree orthogonal polynomials. For a given continuous function f let $L_n(d\alpha, f)$ be the Lagrange interpolating polynomial of degree $n-1$ which agrees with f at $x_{kn}(d\alpha)$, $k=1, 2,..., n$.

THEOREM. *Let* $\log \alpha'(\cos \Theta) \in L^1$, $1 \leqslant p_0 < \infty$ *and* $u(\geqslant 0) \in L^1$. *Suppose that*

$$\int_{-1}^{1} [\alpha'(t)(1-t^2)^{1/2}]^{-p/2}\, u(t)\, dt = \infty \tag{2}$$

* This paper is based upon research supported by the National Science Foundation under Grant MCS-83-00882.

0021-9045/85 $3.00

for every $p > p_0$. *Then there exists a continuous function* f *such that*

$$\limsup_{n \to \infty} \int_{-1}^{1} |L_n(d\alpha, f, t)|^p \, u(t) \, dt = \infty \tag{3}$$

for every $p > p_0$.

COROLLARY (Solution of Turán's Problem). *If* $\log \alpha'(\cos \Theta) \in L^1$ *and*

$$\int_{-1}^{1} (1-t^2)^{-p/4} \alpha'(t)^{1-p/2} \, dt = \infty$$

for every $p > 2$ *then there exists a continuous function* f *such that*

$$\limsup_{n \to \infty} \int_{-1}^{1} |L_n(d\alpha, f, t)|^p \, \alpha'(t) \, dt = \infty$$

for every $p > 2$.

Let us point out that according to Erdös and Turán's celebrated result [1], $\lim L_n(d\alpha, f) = f$ in $L^2_{d\alpha}$ for every continuous function f. Our Theorem easily follows from our results in [3, Chap. 10] and the next technical

LEMMA. *Let* D *be a Banach space with norm* $\|\cdot\|$ *and let* $\{B_p\}_{p_0 < p \leqslant \infty}$ *be a collection of Banach spaces* B_p *with norm* $\|\cdot\|_p$ *such that* $B_p \subset B_q$ *for* $p > q$ *and* $\|b\|_q \leqslant \|b\|_p$ *if* $q < p$ *and* $b \in B_p$. *Let* $\{L_n\}_{n=1}^{\infty}$ *be a sequence of bounded linear operators defined on* D *with values in* B_∞ *such that*

$$\lim_{n \to \infty} \sup_{\|f\| \leqslant 1} \|L_n(f)\|_p = \infty \tag{4}$$

holds for every $p_0 < p \leqslant \infty$. *Then there exists* $f \in D$ *such that*

$$\limsup_{n \to \infty} \|L_n(f)\|_p = \infty \tag{5}$$

for every $p_0 < p \leqslant \infty$.

Proof. We construct f in (5) as follows. First we take $p_1 = p_0 + 1$. Then by (4) and the uniform boundedness principle (UBP) [2, p. 26] there exists $f_1 \in D$ such that $\|f_1\| = 1$ and $\limsup \|L_n(f_1)\|_{p_1} = \infty$. If $\limsup \|L_n(f_1)\|_p = \infty$ for every $p_0 < p < p_1$ then we set $f = f_1$ and (5) is proved. Otherwise, there exists $p_2 < p_1$ such that $p_0 < p_2 < p_0 + 2^{-1}$ and $\limsup \|L_n(f_1)\|_{p_2} < \infty$. With this choice of p_2 we can apply (4) and UBP to find $f_2 \in D$ such that $\|f_2\| = 1$ and $\limsup \|L_n(f_2)\|_{p_2} = \infty$. If $\limsup \|L_n(f_2)\|_p = \infty$ for every $p_0 < p < p_2$ then again (5) is proved with $f = f_2$. Otherwise, there exists $p_3 < p_2$ such that $p_0 < p_3 < p_0 + 3^{-1}$ and limsup

$\|L_n(f_2)\|_{p_3} < \infty$. Now we continue this process and either we find f satisfying (5) or we construct two infinite sequences $\{p_k\}_{k=1}^{\infty}$ and $\{f_k\}_{k=1}^{\infty}$ such that

$$p_1 > p_2 > p_3 > \cdots > p_0, \qquad p_0 < p_k < 2 + k^{-1},$$

$f_k \in D$, $\|f_k\| = 1$, $k = 1, 2, \ldots$,

$$\limsup_{n \to \infty} \|L_n(f_k)\|_{p_j} = \infty, \qquad 1 \leqslant j \leqslant k,$$

and

$$\limsup_{n \to \infty} \|L_n(f_k)\|_{p_j} < \infty, \qquad j > k.$$

Assuming that the latter possibility occurs and introducing the notation

$$A_{kj} = \sup_n \|L_n(f_k)\|_{p_j}, \qquad 1 \leqslant k < j,$$

we can inductively define two sequences $\{\varepsilon_k\}_{k=1}^{\infty}$ and $\{n_k\}_{k=1}^{\infty}$ such that $\varepsilon_1 = \frac{1}{2}$, $0 < \varepsilon_{k+1} \leqslant \frac{1}{2}\varepsilon_k$, n_k's are integers, $1 \leqslant n_1 < n_2 < n_3 < \cdots$,

$$\varepsilon_k \|L_{n_k}(f_k)\|_{p_k} \geqslant k + 2 + \sum_{l=1}^{k-1} \varepsilon_l A_{lk}$$

and

$$\sup_{\|f\| \leqslant 1} \|L_{n_k}(f)\|_{p_k} \leqslant \varepsilon_{k+1}^{-1}.$$

Let f be defined by

$$f = \sum_{k=1}^{\infty} \varepsilon_k f_k.$$

Then $\|f\| \leqslant 1$ and for any given $p > p_0$ we have $p > p_k$ for sufficiently large values of k so that

$$\begin{aligned} \|L_{n_k}(f)\|_p &\geqslant \|L_{n_k}(f)\|_{p_k} \geqslant \varepsilon_k \|L_{n_k}(f_k)\|_{p_k} \\ &\quad - \sum_{l=1}^{k-1} \varepsilon_l \|L_{n_k}(f_l)\|_{p_k} - \sum_{l=k+1}^{\infty} \varepsilon_l \sup_{\|f\| \leqslant 1} \|L_{n_k}(f)\|_{p_k} \\ &\geqslant k + 2 + \sum_{l=1}^{k-1} \varepsilon_l A_{lk} - \sum_{l=1}^{k-1} \varepsilon_l A_{lk} - 2 = k \end{aligned}$$

holds for $k \geqslant k_0$ and thus (5) is satisfied.

Proof of the Theorem. Assuming without loss of generality that $\int u = 1$ we can set $D = C[-1, 1]$, $B_p = L_u^p[-1, 1]$ and $L_n(f) = L_n(d\alpha, f)$. By Theorem 10.15 [3, p. 180] (2) implies (4) and thus by (5) the Theorem is proved.

Using Theorem 10.19 [3, p. 182] one can prove a variant of the Lemma valid for L^p spaces with $0 < p < 1$ and that would extend our Theorem for the case when $0 < p_0 < \infty$. Applying Theorem 10.16 [3, p. 181] one can produce versions of our Theorem where the condition $\log \alpha'(\cos \Theta) \in L^1$ is replaced by other requirements. It is also easy to see that $n \to \infty$ in (2) could be weakened to $n_j \to \infty$ where $\{n_j\}$ is any given increasing sequence of integer. We let the reader fill in the missing links.

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